

A Study of Teleportation and Super Dense Coding capacity in Remote Entanglement Distribution

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In this work we consider a quantum network consisting of nodes and entangled states connecting the nodes. In every node there is a single player. The players at the intermediate nodes carry out measurements to produce an entangled state between the initial and final node. Here we address the problem that how much classical as well as quantum information can be sent from initial node to final node. In this context, we present strong theorems which state that how the teleportation capability of this remotely prepared state is linked up with the fidelities of teleportation of the resource states. Similarly, we analyze the super dense coding capacity of this remotely prepared state in terms of the capacities of the resource entangled states. However, we first obtain the relations involving the amount of entanglement of the resource states with the final state in terms of two different measures of entanglement namely concurrence and entanglement entropy. These relations are quite similar to the bounds obtained in reference [20, 21]. More specifically, in an arbitrary quantum network when two nodes are not connected, our result shows how much information, both quantum and classical can be transmitted between these nodes. We show that the amount of transferable information depends on the capacities of the inter connecting entangled resources. These results have a tremendous future application in the context of determining the optimal path in a quantum network to send the maximal possible information.

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I. INTRODUCTION

For a long time, the concept of entanglement [1] was more of philosophical interest rather than a scientific discovery. However for the last two decades with the advent of quantum information processing protocols, entanglement is now widely recognized as a powerful tool for implementing tasks that cannot be performed using classical means. It had been seen that quantum entanglement plays a pivotal role in large variety of information processing tasks such as teleportation [2], super dense coding [3], remote state preparation [4] and key generation [5, 6]. There are two distinct directions of research involving entanglement. The first one is to find proper criteria of detecting entanglement while the other one is to have a good entanglement measure to quantify the amount of entanglement of a given state. For a pure state the von neumann entropy of the reduced density matrix actually quantifies the amount of entanglement present in the system. This is also known as the entanglement entropy of the given entangled state. Out of various entanglement measures, the measure concurrence is the subject of intense research [7–14]. Concurrence was originally derived from the entanglement of formation (EOF) which is used to compute the amount of entanglement for pure states in two-qubit systems [7]. Since entanglement of formation is a monotonically increasing function of the concurrence, the concurrence itself can also be regarded as an entanglement measure. A natural way of extending the concept of concurrence for two-qubit mixed state is done by convex roof construction [8]. Afterwards, it had been extended to arbitrary but finite-dimensional bipartite as well as

multiparty systems for both pure and mixed states [11, 13]. In an important as well as relevant work, the authors have found that the nature of entanglement present in a multiqubit system is monogamous in nature. This physical phenomenon is accredited by a relation involving the concurrences of the n -multiqubit system and its subsystems [19]. It was shown that concurrence also plays a major in various protocols of RED [20, 21].

Teleportation and super dense coding are regarded as the two major achievements of quantum information theory [22, 23]. In each of these two information processing protocols entanglement plays a pivotal role in execution of them. In short, teleportation and super dense coding is all about sending quantum and classical information through a quantum resource. In classical information theory we have seen networking plays a key role in sending information from one node to a distant node. In quantum networking one of the most challenging problem is to know how much classical and quantum information one can send from one node to a distant node which are not initially entangled. In this work we claim to provide a solution to this problem by finding out the teleportation fidelity and super dense coding capacity of the remotely prepared state in terms of teleportation fidelities and super dense coding capacities of the resource states. In particular, we find out these relations in the context of the entanglement distribution between distant nodes by the standard swapping of the entangled resource states. But before that we find such relations involving the amount of entanglement of the resource states with the final state in terms of two different measures of entanglement namely concurrence and entanglement entropy.

These relations are quite similar to the bounds obtained in reference [20, 21], where the authors have obtained the bounds in terms of G-concurrence. Then by using these relations we establish the relations involving the teleportation fidelity and superdense coding capacity of the entangled channels that can be produced in terms of RED protocols.

The organization of the paper is as follows. In section II, we start with two pure entangled resource states shared by three parties and obtain the relations involving the concurrences of the resource states with the final state obtained in the process of RED by swapping. In section III, we extend our results where we have more than two resource states and three parties. In section IV, we once again start with two pure entangled resource states shared by three parties and obtain the relations connecting von-neumann entropy of the states present before and after swapping. In section V, we provide strong results connecting the teleportation fidelity and superdense coding capability of the resource states with the state engineered by the process of entanglement swapping. Finally we conclude in section VI.

II. RELATIONS ON CONCURRENCE FOR TRIPARTITE SITUATION IN REMOTE ENTANGLED DISTRIBUTION (RED)

In this section, we consider the most simplest situation where Alice and Bob share a pure entangled state $|\psi\rangle_{12}$ between them. Similarly, Bob and Charlie also share another entangled state $|\psi\rangle_{23}$ between them. This is equivalent of saying, we have entanglement between the nodes 1 and 2 as well as between the nodes 2 and 3. Our aim is to establish the entanglement between the remote nodes 1 and 3 which are not initially entangled. We adopt the procedure of entanglement swapping to carry out the remote entanglement distribution between the nodes 1 and 3. In order to swap the entanglement, Bob carries out measurement on his qubits which are at the node 2. Based on Bob's measurement outcomes $|\phi^{rh}\rangle$ ($r, h = 0, 1$), Alice and Charlie end up with states $|\chi^{rh}\rangle_{13}$ respectively. Interestingly, we find an important relationship between the concurrences of the entangled states before and after swapping. The most remarkable aspect of this relationship is that this tells us about the amount of entanglement that can be created in a remote entanglement distribution (RED) via swapping.

Relations on Concurrence for Two Qubit Pure States

In this subsection we start with resource entangled states in $2 \otimes 2$ dimensions. These states are given by

$$|\psi\rangle_{12} = \sum_{i,j} a_{ij} |ij\rangle, \quad (1)$$

$$|\psi\rangle_{23} = \sum_{p,q} b_{pq} |pq\rangle, \quad (2)$$

respectively. Here, $a_{ij}, b_{pq} \in \mathcal{C}$ ($i, j, p, q = 0, 1$) are the probability amplitudes satisfying the normalization conditions $a_{00}^2 + a_{01}^2 + a_{10}^2 + a_{11}^2 = 1$ and $b_{00}^2 + b_{01}^2 + b_{10}^2 + b_{11}^2 = 1$.

We consider a situation, where we take into account a general measurement strategy. Here, Bob carries out measurement in a non maximally entangled basis given by the basis vectors, $|\phi_G^{00}\rangle = \frac{1}{\sqrt{1+n^2}}(|00\rangle + n|11\rangle)$, $|\phi_G^{10}\rangle = \frac{1}{\sqrt{1+n^2}}(n|00\rangle - |11\rangle)$, $|\phi_G^{01}\rangle = \frac{1}{\sqrt{1+m^2}}(|01\rangle - m|10\rangle)$ and $|\phi_G^{11}\rangle = \frac{1}{\sqrt{1+m^2}}(m|01\rangle - |10\rangle)$. Here the indices $n, m \in \mathcal{C}$ are the entangling parameters and they lie between 0 and 1 i.e., $0 \leq (n, m) \leq 1$. Now according to general measurements done by Bob on his qubits, we have four possible states between the nodes 1 and 3 at Alice and Charlie's locations respectively. These four possible states based on Bob's measurement outcomes $|\phi_G^{rh}\rangle$ ($r, h = 0, 1$) are given by,

$$|\chi^{rh}\rangle_{13} = \frac{1}{\sqrt{M_{rh}}} \sum_{i,q=0}^1 \left(\sum_{j=0}^1 e^{-I\pi r j} R_j^{rh} a_{ij} b_{j \oplus h, q} \right) |iq\rangle_{13}. \quad (3)$$

The modulo sum $j \oplus h$ represents the sum of j and h modulo 2 and the normalization factors are given by $M_{rh} = \sum_{i,q=0}^1 \left(\sum_{j=0}^1 e^{-I\pi r j} R_j^{rh} a_{ij} b_{j \oplus h, q} \right)^2$. The coefficients R_j^{rh} in equation (3) are defined as

$$R_j^{rh} = \begin{cases} n & \text{if } (r, h, j) = (0, 0, 1) \text{ or } (1, 0, 0), \\ m & \text{if } (r, h, j) = (0, 1, 1) \text{ or } (1, 1, 0), \\ 1 & \text{otherwise.} \end{cases} \quad (4)$$

Interestingly, here we obtain an important relation between the concurrences of the initial and final states,

$$C(|\chi^{rh}\rangle_{13}) = \frac{F_{rh}}{2M_{rh}} C(|\psi\rangle_{12}) C(|\psi\rangle_{23}), \quad (5)$$

where the coefficients F_{rh} are given by,

$$F_{rh} = \begin{cases} n & \text{if } (r, h) = (0, 0) \text{ or } (1, 0), \\ m & \text{if } (r, h) = (0, 1) \text{ or } (1, 1). \end{cases} \quad (6)$$

This relation (5) shows that we can always determine the amount of entanglement to be created between the unentangled nodes depending upon the choice of the resource states.

Relations on Concurrence for Two Qudit Pure States

In this subsection we extend our result to the situation where we have entangled states in $d \otimes d$ dimension instead of states in $2 \otimes 2$ dimension. If we know the state properly then we can always rewrite it in the Schmidt decomposed [25] form. If we have a pure two-qudit state in the form $|\psi\rangle = \sum_{i,j=0}^{d-1} a_{ij}|ij\rangle$ where $\sum_{i,j=0}^{d-1} a_{ij}^2 = 1$, then the Schmidt decomposition form for this state will be

$$|\psi\rangle = \sum_{\tilde{i}=0}^{d-1} \lambda_{\tilde{i}} |\tilde{i}\tilde{i}\rangle, \quad (7)$$

where $\sum_{i=0}^{d-1} \lambda_i^2 = 1$ and λ_i are real and non-negative, and $\{|i\rangle\}$ is an orthonormal basis of the corresponding Hilbert space. The concurrence for two-qudit state $|\psi\rangle$ can be written in the form [8, 9]

$$C(|\psi\rangle) = \sqrt{\frac{d}{d-1}(1 - I_1)}, \quad (8)$$

where, $I_1 = \text{tr}[\rho_A^2] = \text{tr}[\rho_B^2] = \sum_{i=0}^{d-1} \lambda_i^4$. Here ρ_A and ρ_B are reduced density matrices for sub-systems. The term $1 - I_1$ in equation (8) can be simplified as,

$$\begin{aligned} 1 - I_1 &= 1 - \sum_{i=0}^{d-1} \lambda_i^4 \\ &= \left(\sum_{i=0}^{d-1} \lambda_i^2 \right)^2 - \sum_{i=0}^{d-1} \lambda_i^4 \\ &= 2 \sum_{i,j=0(i<j)}^{d-1} \lambda_i^2 \lambda_j^2. \end{aligned} \quad (9)$$

Hence, the simplified version of concurrence of a two-qudit pure state is given by,

$$C(|\psi\rangle) = \sqrt{\frac{2d}{d-1} \left(\sum_{i,j=0(i<j)}^{d-1} \lambda_i^2 \lambda_j^2 \right)}. \quad (10)$$

For $d = 2$, this equation reduces to $C = 2 |\lambda_0 \lambda_1|$.

Let us consider a two-qudit pure state shared by parties Alice and Bob $|\psi\rangle_{12} = \sum_{i=0}^{d-1} \lambda_i |ii\rangle$ and Bob and Charlie

shares the pure two-qudit state $|\psi\rangle_{23} = \sum_{j=0}^{d-1} \mu_j |jj\rangle$, where

$\sum_{i=0}^{d-1} \lambda_i^2 = 1, \sum_{j=0}^{d-1} \mu_j^2 = 1$. In other words, $|\psi\rangle_{12}$ is the entanglement shared between the nodes 1 and 2, whereas $|\psi\rangle_{23}$ is the entanglement between the nodes 2 and 3. Now Bob carries out Bell measurements on his qudits. These basis vectors on which the Bell measurements are carried out are given by,

$$|\phi^{rh}\rangle = \frac{1}{\sqrt{d}} \sum_{t=0}^{d-1} e^{\frac{2\pi i r t}{d}} |t\rangle |t \oplus h\rangle, \quad (11)$$

where $t \oplus h$ means the sum of t and h modulo d . The indices r and h can take integer values between 0 and $d-1$. We can revert the above equation to obtain

$$|ij\rangle = \frac{1}{\sqrt{d}} \sum_{r,h=0}^{d-1} e^{\frac{-2\pi i r j}{d}} \delta_{i,i \oplus h} |\phi^{rh}\rangle. \quad (12)$$

Hence, the combined state of Alice, Bob and Charlie is

$$\begin{aligned} |\Phi\rangle_{1223} &= |\psi\rangle_{12} \otimes |\psi\rangle_{23} \\ &= \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \lambda_i \mu_j |ij\rangle_{13} |ij\rangle_{22} \\ &= \frac{1}{\sqrt{d}} \sum_{i,j}^{d-1} \sum_{r,h}^{d-1} e^{\frac{-2\pi i r j}{d}} \lambda_i \mu_j |ij\rangle_{13} \delta_{j,i \oplus h} |\phi^{rh}\rangle_{22} \\ &= \frac{1}{\sqrt{d}} \sum_{i,r,h}^{d-1} e^{\frac{-2\pi i r j}{d}} \lambda_i \mu_{i \oplus h} |i, i \oplus h\rangle_{13} |\phi^{rh}\rangle_{22} \end{aligned} \quad (13)$$

According to measurement outcomes $|\phi^{rh}\rangle_{22}$ on Bob's side, the states that are created between the nodes 1 and 3 are given by

$$|\chi^{rh}\rangle_{13} = \frac{1}{\sqrt{N_{rh}}} \sum_{i=0}^{d-1} e^{\frac{-2\pi i r j}{d}} \lambda_i \mu_{i \oplus h} |i, i \oplus h\rangle_{13} \quad (14)$$

where $N_{rh} = \sum_{i=0}^{d-1} \lambda_i^2 \mu_{i \oplus h}^2$ are normalization factors.

We can construct unitary operators of the form $U_{st} = \sum_{r=0}^{d-1} e^{\frac{2\pi i s r}{d}} |r\rangle \langle r \oplus t|$ which can transform states in equation

(14) into its diagonal form. Now in order to calculate concurrence of the states in equation (14), first we need to simplify the term $1 - I_1$ defined in equation (8)), for this state in (14). On simplification we have,

$$\begin{aligned} 1 - I_1 &= 1 - \frac{1}{N_{rh}^2} \sum_{i=0}^{d-1} \lambda_i^4 \mu_{i \oplus h}^4 \\ &= \frac{1}{N_{rh}^2} \left[\left(\sum_{i=0}^{d-1} \lambda_i^2 \mu_{i \oplus h}^2 \right)^2 - \sum_{i=0}^{d-1} \lambda_i^4 \mu_{i \oplus h}^4 \right] \\ &= \frac{2}{N_{rh}^2} \sum_{i,f=0(i < f)}^{d-1} (\lambda_i^2 \lambda_f^2) (\mu_{i \oplus h}^2 \mu_{f \oplus h}^2). \end{aligned} \quad (15)$$

Hence, the concurrence of the final two-qudit state is given by,

$$\begin{aligned} C(|\chi^{rh}\rangle_{13}) &= \\ \frac{1}{N_{rh}} \sqrt{\frac{2d}{d-1} \left(\sum_{i < f}^{d-1} (\lambda_i^2 \lambda_f^2) (\mu_{i \oplus h}^2 \mu_{f \oplus h}^2) \right)}. \end{aligned} \quad (16)$$

To understand the terms in equation (16) we have to split it in the following way

$$\begin{aligned} \sum_{i < f}^{d-1} (\lambda_i^2 \lambda_f^2) (\mu_{i \oplus h}^2 \mu_{f \oplus h}^2) &= \sum_{i < f}^{d-1} \lambda_i^2 \lambda_f^2 \sum_{i < f}^{d-1} \mu_{i \oplus h}^2 \mu_{f \oplus h}^2 \\ &\quad - \sum_{i < f}^{d-1} (\lambda_i^2 \lambda_f^2 \sum_{l < m}^{d-1} \Theta_{lm}^{if} \mu_{l \oplus h}^2 \mu_{m \oplus h}^2), \end{aligned} \quad (17)$$

where function Θ_{lm}^{if} is defined as

$$\Theta_{lm}^{if} = \begin{cases} 1 & \text{if } (l, m) \neq (i, f) \text{ for } d \geq 3, \\ 0 & \text{if } (l, m) = (i, f) \text{ or } d \leq 2. \end{cases} \quad (18)$$

It is evident that in case of $d \otimes d$ dimensions, we have no direct relationship as we have obtained in the multiqubit case. However we consider a special situation where we have only two non vanishing Schmidt coefficients, then we have the concurrence of the state $|\psi\rangle$ as

$$C_{ij}(|\psi\rangle) = \sqrt{\frac{2d}{d-1}} (\lambda_i \lambda_j). \quad (19)$$

The C_{ij} are the concurrences of $|\psi\rangle$, when two of the Schmidt coefficients are present only. Then we have the relation with the concurrences of the initial and final entangled states as

$$C^2(|\chi^{rh}\rangle_{13}) = \frac{(d-1)}{2dN_{rh}^2} [C^2(|\psi\rangle_{12}) \cdot C^2(|\psi\rangle_{23}) - K_d^h], \quad (20)$$

where $K_d^h = \sum_{i < f}^{d-1} (C_{if}^2(|\psi\rangle_{12}) \cdot \sum_{l < m}^{d-1} \Theta_{lm}^{if} C_{l \oplus h, m \oplus h}^2(|\psi\rangle_{23}))$ is a term that depends on dimension d and $K_2^q = 0$ only when $d = 2$. Hence for $d = 2$, equation (20) becomes

$$C(|\chi^{pq}\rangle_{13}) = \frac{1}{2N_{rh}} C(|\psi\rangle_{12}) C(|\psi\rangle_{23}). \quad (21)$$

This relation involving the concurrences reflects that the amount of entanglement that can be created between the remote nodes is solely a function of the amount of entanglement of the resource states.

III. RELATIONS ON CONCURRENCE FOR MULTIPARTY SITUATION IN REMOTE ENTANGLED DISTRIBUTION (RED)

In this section we study a more general situation where we have more numbers of initial entangled states rather than a pair. In other words we are having more than three nodes to begin. The consecutive nodes are entangled. We need to establish the entanglement between the initial and the final nodes which are remotely located. We consider the process of entanglement swapping as a technique for the remote entanglement distribution (RED). Here in this section we extend these relations obtained for entangled qubits in previous section in a more general situation where we have more than three nodes.

Let us assume that we have $(g+1)$ entangled states in $2 \otimes 2$ dimensions with $g+2$ nodes. In order to obtain an entangled state between 1st and last node we carry out g number of entanglement swappings. We separately study two different types of measurement strategies in the entire swapping procedure. First of all we consider the case where we carry out simultaneous measurement in a non maximally entangled basis in each of these intermediate nodes to obtain an entangled state between the qubits in the first and the last node. Secondly, we consider sequential measurements to create successive entanglements between the nodes $(1, 3)$, $(1, 4)$ and finally between the nodes $(1, g+2)$. In each of these cases we obtain the extension of the relationships involving the concurrences of initial and final entangled states.

Simultaneous Measurement:

In this subsection we start with $(g+1)$ entangled states in the most general form, $|\psi\rangle = \sum_{i_k, j_k=0} a_{i_k j_k} |i_k j_k\rangle$, where k denotes the index for the number of entangled states and varies from 0 to g . Here for a fixed k , $a_{i_k j_k}$ denotes the corresponding coefficients of the given entangled state. Then we

create an entangled state between the qubits at the nodes 1 and $g + 2$ by entanglement swapping. In other words we carry out simultaneous measurements $M1, M2, \dots, Mg$ at the nodes 2, 3, 4, ..., $g + 1$ respectively to obtain an entangled state between the qubits at the nodes 1 and $g + 2$ [see Fig.(1)].

After evaluating the concurrences for the initial states and final state we find them to be related by,

$$C(|\chi^{r_1 h_1, r_2 h_2, \dots, r_g h_g}\rangle_{1(g+2)}) = \frac{\prod_{i=1}^g F_{r_i h_i}}{2^g M_{r_1 h_1, r_2 h_2, \dots, r_g h_g}} C(|\psi\rangle_{12}) C(|\psi\rangle_{23}) \dots C(|\psi\rangle_{(g+1)(g+2)}). \quad (22)$$

Here the indices r and h take the values 0 and 1 and the subscript $i (= 1, 2, \dots, g)$ denotes the number of measurements that have taken place. The normalization factors are given by $M_{r_1 h_1, r_2 h_2, \dots, r_g h_g} =$

$$\sum_{i_0, j_g=0}^1 \left(\sum_{j_1, j_2, \dots, j_{g-1}=0}^1 e^{-I\pi r_1 j_0} e^{-I\pi r_2 j_1} \dots e^{-I\pi r_g j_{g-1}} R_{j_0}^{r_1 h_1} R_{j_1}^{r_2 h_2} \dots R_{j_{g-1}}^{r_g h_g} a_{i_0 j_0} a_{j_0 \oplus h_1 j_1} a_{j_1 \oplus h_2 j_2} \dots a_{j_{g-1} \oplus h_g j_g} \right)^2. \quad \text{The superscripts } r_i, h_i \in [0, 1], i = 1, 2, \dots, g \text{ comes from the measurement of } g \text{ parties (i.e., if their measurement results are } |\phi^{r_1 h_1}\rangle \otimes |\phi^{r_2 h_2}\rangle \otimes \dots \otimes |\phi^{r_g h_g}\rangle = \bigotimes_{i=1}^g |\phi^{r_i h_i}\rangle \text{). The resultant states obtained after swapping are given by,}$$

$$|\chi^{r_1 h_1, r_2 h_2, \dots, r_g h_g}\rangle_{1(g+2)} = \frac{1}{\sqrt{M_{r_1 h_1, r_2 h_2, \dots, r_g h_g}}} \sum_{i_0, j_g=0}^1 \left(\sum_{j_1, j_2, \dots, j_{g-1}=0}^1 e^{-I\pi r_1 j_0} e^{-I\pi r_2 j_1} \dots e^{-I\pi r_g j_{g-1}} R_{j_0}^{r_1 h_1} R_{j_1}^{r_2 h_2} \dots R_{j_{g-1}}^{r_g h_g} a_{i_0 j_0} a_{j_0 \oplus h_1 j_1} a_{j_1 \oplus h_2 j_2} \dots a_{j_{g-1} \oplus h_g j_g} \right) |i_0, j_g\rangle_{1(g+2)}. \quad (23)$$

The coefficients, $F_{r_i h_i}$ and $R_{j_i}^{r_i h_i}$ are defined as

$$F_{r_i h_i} = \begin{cases} n & \text{if } (r_i, h_i) = (0, 0) \text{ or } (1, 0), \\ m & \text{if } (r_i, h_i) = (0, 1) \text{ or } (1, 1) \end{cases} \quad (24)$$

and

$$R_{j_i}^{r_i h_i} = \begin{cases} n & \text{if } (r_i, h_i, j_i) = (0, 0, 1) \text{ or } (1, 0, 0), \\ m & \text{if } (r_i, h_i, j_i) = (0, 1, 1) \text{ or } (1, 1, 0), \\ 1 & \text{otherwise.} \end{cases} \quad (25)$$

Sequential Measurement:

Next we consider a different type of measurement strategy, where we carry out measurements $M1, M2, \dots, Mg$ one after the other to create successive entanglement between the

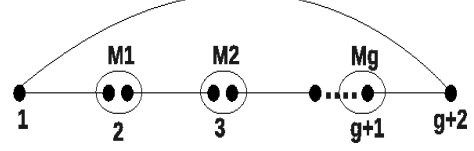


FIG. 1: In this Figure entanglement swapping is done with simultaneous measurements. Here g number of measurements $M1, M2, \dots, Mg$ are carried out simultaneously at the nodes (2, 3, 4, ..., $g + 1$) to obtain an entangled state between the first and last node (i.e., 1, $g + 2$)).

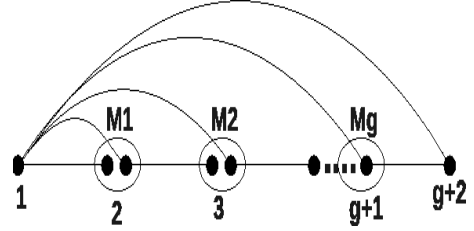


FIG. 2: In this Figure entanglement swapping is done with sequential measurements. Here g number of measurements $M1, M2, \dots, Mg$ are carried out sequentially to obtain successive entangle states between the nodes (1, 3), (1, 4), ..., and finally between the nodes (1, $g + 2$).

pair of nodes (1, 3), (1, 4), ..., (1, $g + 2$) respectively [see Fig.(2)]. Even in case of sequential measurements, we also see that there exists the same relationship between the concurrence of the initial and final states.

Quite similar to the previous situation, here also the concurrence of the initial and final states is to be related by the same relationship,

$$C(|\chi^{r_1 h_1, r_2 h_2, \dots, r_g h_g}\rangle_{1(g+2)}) = \frac{\prod_{i=1}^g F_{r_i h_i}}{2^g M_{r_1 h_1, r_2 h_2, \dots, r_g h_g}} C(|\psi\rangle_{12}) C(|\psi\rangle_{23}) \dots C(|\psi\rangle_{(g+1)(g+2)}), \quad (26)$$

where the indices have their usual significances.

IV. RELATIONS ON ENTANGLEMENT ENTROPY FOR TRIPARTITE SITUATION IN REMOTE ENTANGLEMENT DISTRIBUTION (RED)

In this section we obtain the relations relating the entanglement entropy of the resource states with the final state obtained after swapping in the context of remote entanglement distribution (RED). Entropic measure of entanglement of a bipartite state ρ_{AB} is defined as

$$E(\rho_{AB}) = -\text{tr}[\rho' \log_2 \rho'], \quad (27)$$

where, $\rho' = \text{tr}_A[\rho_{AB}] = \text{tr}_B[\rho_{AB}]$. For $d \otimes d$ dimension it is bounded by

$$0 \leq E(\rho_{AB}) \leq \log_2 d. \quad (28)$$

Here we obtain the relations involving the entanglement entropy for the simplest situation involving three parties. Consider two states in $d \otimes d$ dimensions shared by the parties Alice and Bob and Bob and Charlie respectively. These states are $|\psi\rangle_{12} = \sum_{i=0}^{d-1} \lambda_i |ii\rangle$ and $|\psi\rangle_{23} = \sum_{j=0}^{d-1} \mu_j |jj\rangle$ between the nodes 1 and 2 and 2 and 3 respectively. Entanglement entropy of these two states are given by,

$$\begin{aligned} E(|\psi\rangle_{12}) &= -\sum_i \lambda_i^2 \log_2 \lambda_i^2, \\ E(|\psi\rangle_{23}) &= -\sum_j \mu_j^2 \log_2 \mu_j^2. \end{aligned} \quad (29)$$

Then Bob carries out the generalized Bell state measurement on his qubits at the node 2 in the basis

$$|\phi^{rh}\rangle = \frac{1}{\sqrt{d}} \sum_{t=0}^{d-1} e^{\frac{2\pi i r t}{d}} |t\rangle |t \oplus h\rangle, \quad (30)$$

where $t \oplus h$ means the sum of t and h modulo d . The indices r and h can take integer values between 0 and $d-1$. According to measurement outcomes $|\phi^{rh}\rangle_{23}$ on Bob's side, the resultant entangled pairs generated between the nodes 1 and 3 are given by,

$$|\chi^{rh}\rangle_{13} = \frac{1}{\sqrt{N_{rh}}} \sum_{i=0}^{d-1} e^{\frac{-2\pi i r i}{d}} \lambda_i \mu_{i \oplus h} |i, i \oplus h\rangle_{13} \quad (31)$$

where $N_{rh} = \sum_{i=0}^{d-1} \lambda_i^2 \mu_{i \oplus h}^2$ are normalization factors.

Here we construct unitary operators of the form $U_{st} = \sum_{r=0}^{d-1} e^{\frac{2\pi i s r}{d}} |r\rangle \langle r \oplus t|$ which transform states in Eqn.(31) into

its diagonal form. The entanglement entropy of these states are given by,

$$E(|\chi^{rh}\rangle_{13}) = -\frac{1}{N_{rh}} \sum_i \lambda_i^2 \mu_{i \oplus h}^2 \log_2 \left[\frac{\lambda_i^2 \mu_{i \oplus h}^2}{N_{rh}} \right]. \quad (32)$$

Here we discuss three important cases that arise because of the particular choice of Schmidt coefficients,

Case I:

First of all we consider the case when both the states are maximally entangled, i.e λ_i, μ_j are all equal to $\frac{1}{\sqrt{d}}$, then the entanglement entropy of the initial and final states are related by,

$$E(|\chi^{rh}\rangle_{13}) = E(|\psi\rangle_{12}) = E(|\psi\rangle_{23}) = \log_2 d. \quad (33)$$

Case II:

Here we consider the situation where one of the resource states is maximally entangled instead of both being maximally entangled.

a) If we assume $\lambda_i = \frac{1}{\sqrt{d}}, \mu_j \neq \frac{1}{\sqrt{d}}$, then the entanglement entropy of the resource states are given by $E(|\psi\rangle_{12}) = \log_2 d$. but $E(|\psi\rangle_{23}) = -\sum_j \mu_j^2 \log_2 \mu_j^2 < \log_2 d$. Therefore, the entanglement entropy of the swapped state is given by,

$$\begin{aligned} E(|\chi^{rh}\rangle_{13}) &= -d \sum_i \frac{1}{d} \mu_{i \oplus h}^2 \log_2 [\mu_{i \oplus h}^2] \\ &= -\sum_j \mu_j^2 \log_2 [\mu_j^2] = E(|\psi\rangle_{23}). \end{aligned} \quad (34)$$

b) If we assume $\lambda_i \neq \frac{1}{\sqrt{d}}, \mu_j = \frac{1}{\sqrt{d}}$, i.e the entropy of the resource states are $E(|\psi\rangle_{12}) = \sum_i \lambda_i^2 \log_2 \lambda_i^2 < \log_2 d$, $E(|\psi\rangle_{23}) = \log_2 d$, then the entanglement entropy of the swapped state is given by,

$$E(|\chi^{rh}\rangle_{13}) = -\sum_i \lambda_i^2 \log_2 [\lambda_i^2] = E(|\psi\rangle_{12}). \quad (35)$$

Case III:

Finally, we consider the case when neither of the resource entangled states are maximally entangled. In other words, we assume $\lambda_i \neq \frac{1}{\sqrt{d}}, \mu_j \neq \frac{1}{\sqrt{d}}$, i.e the entropy of the resource states are $E(|\psi\rangle_{12}) = \sum_i \lambda_i^2 \log_2 \lambda_i^2 < \log_2 d$, $E(|\psi\rangle_{23}) = -\sum_j \mu_j^2 \log_2 \mu_j^2 < \log_2 d$, then the entanglement entropy of the swapped state is given by,

$$E(|\chi^{rh}\rangle_{13}) < \max[E(|\psi\rangle_{12}), E(|\psi\rangle_{23})]. \quad (36)$$

V RELATIONS ON TELEPORTATION FIDELITY AND SUPERDENSE CODING CAPACITY IN REMOTE ENTANGLED DISTRIBUTION (RED)

Quantum teleportation and quantum super dense coding are typical information processing tasks where at present there is intense activity in extending the experimental frontiers [19]. In this section we have obtained relation for teleportation fidelity and super dense coding capacity of a remotely prepared entangled states with that of the resource states. These relations are very much important and relevant in the context of quantum networking. A quantum network, is a collection of nodes interconnected by entangled states that allow sharing of resources and information. Here we consider quantum network consisting of nodes in sequence and pure entangled states connecting consecutive pair of nodes. One might ask an important question in this context that in an quantum network what is the amount of quantum information and classical information one can send between the initial and final nodes. In other words is there any relation connecting the information processing capabilities of the resource states with the final state. We see that indeed there is certain relation, which determines the amount of information one can send from the initial and final node after creating an entangled state between the initial and final node through the process of remote entanglement distribution (RED). In particular we prove a strong theorem which states the how the teleportation capability of a remotely prepared state is linked up with the fidelity of teleportation of the initial resource states. Similarly, we analyzed the super dense coding capacity of the remotely prepared state in terms of the capacity of the initial entangled states. In other words these analysis both in case of teleportation and super dense coding shows that the amount of information both quantum and classical one can send between two unentangled nodes is dependent on the choice of resource states. This actually paves the way in determining the path in an arbitrary quantum network through which we can send maximal possible quantum information between any two unentangled nodes.

Relations On Teleportation Fidelity In Remote Entangled Distribution (RED)

Quantum teleportation is all about sending a quantum information of all sending quantum information of one party to other with the help of a resource entangled state. It is well known that all pure entangled states $2 \otimes 2$ dimensions are useful for teleportation. However, the situation is not so trivial for mixed entangled states. There arise situation where given an entangled state it can not be used as a resource for teleportation. However after suitable local operation and classical

communication (LOCC) one can always convert an entangled state not useful for teleportation to a state useful for teleportation. For a given mixed state,

$$\rho = \frac{1}{4}(I \otimes I + \sum_i r_i \cdot \sigma_i \otimes I + \sum_i s_i \cdot I \otimes \sigma_i + \sum_{ij} t_{ij} \sigma_i \otimes \sigma_j), \quad (37)$$

the teleportation fidelity measuring the capability of the state ρ to act as a resource for teleportation, is given by,

$$F = \frac{1}{2}[1 + \frac{1}{3}(\sum_i \sqrt{u_i})]. \quad (38)$$

Here in equation (37), $\sigma_i = (\sigma_1, \sigma_2, \sigma_3)$ are the pauli matrices; $r_i = (r_1, r_2, r_3)$, $s_i = (s_1, s_2, s_3)$ are the unit vectors and t_{ij} are the elements of the correlation matrix $T = [t_{ij}]_{3 \times 3}$. The quantities u_i are the eigenvalues of the matrix $U = T^\dagger T$. A quantum state is said to be useful for teleportation when the value of the quantity F is more than the classically achievable limit of fidelity of teleportation, which is $\frac{2}{3}$. The entangled Werner state [24] in $2 \otimes 2$ dimensions is one example of a useful resource for teleportation for a certain range of classical probability of mixing [26]. Other examples, of mixed entangled states as a resource for teleportation are also there [26–28].

In this subsection we investigate how the teleportation fidelities of the resource states are connected with that of the entangled state obtained as a result of entanglement swapping involving the resource states. More precisely in a quantum network we consider a situation where we have various resource states the consecutive nodes. Now, if we want to send the quantum information from the initial node to the final node, one way of doing it is by first creating an entangled state between these two nodes, which is obtained after doing measurements on the intermediate nodes [20, 21]. This is known as entanglement swapping and falls into the broader class of remote entanglement distribution (RED). Once the entangled state is created we can send the quantum information. Here in this subsection we give an important theorems connecting the teleportation fidelities (capacities of sending quantum information) of the resource entangled states with that of the final entangled state. This remarkably tells us about the capacity of a quantum network in sending quantum information between two desired nodes.

Let us begin with very simplistic situation where there are two parties Alice, Bob share an entangled state $|\psi\rangle_{12} = \lambda_0|00\rangle + \lambda_1|11\rangle$ between them, where as Bob and Charlie share another state $|\psi\rangle_{23} = \mu_0|00\rangle + \mu_1|11\rangle$

(where $\lambda_i, \mu_i, (i, j = 0, 1)$ are the Schmidt coefficients, satisfying $\sum_i \lambda_i^2 = 1, \sum_j \mu_j^2 = 1$) with each other [29]. This is equivalent of saying that we have considered the parties and nodes to be synonymous, then $|\psi\rangle_{12}$ and $|\psi\rangle_{23}$ are the respective entangled states between the nodes (1, 2) and (2, 3). We then propose the following theorem connecting the teleportation capability of the resource states with that of the entangled state obtained between the non connected nodes as a result of swapping.

Theorem I: For the initial resource states written in the Schmidt decomposition form $|\psi\rangle_{12} = \lambda_0|00\rangle + \lambda_1|11\rangle$ and $|\psi\rangle_{23} = \mu_0|00\rangle + \mu_1|11\rangle$, where $\lambda_i, \mu_j, (i, j = 0, 1)$ are the Schmidt coefficients (satisfying $\sum_i \lambda_i^2 = 1, \sum_j \mu_j^2 = 1$), the teleportation fidelities of the initial states and final state $|\chi\rangle_{13}$ obtained after the measurement in the general basis $|\phi_G^{00}\rangle = \frac{1}{\sqrt{1+n^2}}(|00\rangle + n|11\rangle), |\phi_G^{10}\rangle = \frac{1}{\sqrt{1+n^2}}(n|00\rangle - |11\rangle), |\phi_G^{01}\rangle = \frac{1}{\sqrt{1+m^2}}(|01\rangle - m|10\rangle)$ and $|\phi_G^{11}\rangle = \frac{1}{\sqrt{1+m^2}}(m|01\rangle - |10\rangle), (0 < n, m < 1)$ are related by, $3F(|\chi^{rh}\rangle_{13}) - 2 = \frac{F_{pq}}{2M_{pq}} [3F(|\psi\rangle_{12}) - 2] [3F(|\psi\rangle_{23}) - 2]$, where F_{pq} is a function of the measurement parameters, M_{pq} are the normalization constants and r, h are the indices to denote the measurement outcomes.

Proof: Here we start with two resource states $|\psi\rangle_{12} = \lambda_0|00\rangle + \lambda_1|11\rangle$ and $|\psi\rangle_{23} = \mu_0|00\rangle + \mu_1|11\rangle$ between the nodes (1, 2) and (2, 3). Then after applying measurement in the general basis $|\phi_G^{00}\rangle = \frac{1}{\sqrt{1+n^2}}(|00\rangle + n|11\rangle), |\phi_G^{10}\rangle = \frac{1}{\sqrt{1+n^2}}(n|00\rangle - |11\rangle), |\phi_G^{01}\rangle = \frac{1}{\sqrt{1+m^2}}(|01\rangle - m|10\rangle)$ and $|\phi_G^{11}\rangle = \frac{1}{\sqrt{1+m^2}}(m|01\rangle - |10\rangle)$ at the node 2, we obtain the entangled states between the nodes (1, 3). These entangled states obtained on basis of the measurement outcomes $|\phi_G^{rh}\rangle (r, h = 0, 1)$ are given by,

$$|\chi^{rh}\rangle_{13} = \frac{1}{\sqrt{M_{rh}}} \sum_{i,q=0}^1 \left(\sum_{j=0}^1 e^{-I\pi rj} R_j^{rh} a_{ij} b_{j \oplus h, q} \right) |iq\rangle_{13}. \quad (39)$$

The modulo sum $j \oplus h$ represents the sum of j and h modulo 2 and the normalization factors are given by $M_{rh} = \sum_{i,q=0}^1 \left(\sum_{j=0}^1 e^{-I\pi rj} R_j^{rh} a_{ij} b_{j \oplus h, q} \right)^2$. Here the coefficients R_l^{rh} are given by,

$$R_j^{rh} = \begin{cases} n & \text{if } (r, h, j) = (0, 0, 1) \text{ or } (1, 0, 0), \\ m & \text{if } (r, h, j) = (0, 1, 1) \text{ or } (1, 1, 0), \\ 1 & \text{otherwise.} \end{cases} \quad (40)$$

The concurrences of all those pure states before and after swapping are given by,

$$C(|\chi^{rh}\rangle_{13}) = \frac{F_{pq}}{2M_{pq}} C(|\psi\rangle_{12}) C(|\psi\rangle_{23}), \quad (41)$$

where the term F_{pq} is defined as

$$F_{pq} = \begin{cases} n & \text{if } (p, q) = (0, 0) \text{ or } (1, 0) \\ m & \text{if } (p, q) = (0, 1) \text{ or } (1, 1). \end{cases}$$

On the other hand the fidelities of the initial and final states are given by,

$$\begin{aligned} F(|\psi\rangle_{12}) &= \frac{2}{3} (1 + \lambda_0 \lambda_1) = \frac{1}{3} (2 + C(|\psi\rangle_{12})) \\ F(|\psi\rangle_{23}) &= \frac{2}{3} (1 + \mu_0 \mu_1) = \frac{1}{3} (2 + C(|\psi\rangle_{23})) \\ F(|\chi^{rh}\rangle_{13}) &= \frac{2}{3} \left(1 + \frac{F_{pq} \lambda_0 \lambda_1 \mu_0 \mu_1}{M_{pq}} \right). \end{aligned} \quad (42)$$

Therefore, after expressing the concurrences in terms of the teleportation fidelities and then substituting in (41) we find these fidelities to be related by,

$$3F(|\chi^{rh}\rangle_{13}) - 2 = \frac{F_{pq}}{2M_{pq}} [3F(|\psi\rangle_{12}) - 2] [3F(|\psi\rangle_{23}) - 2] \quad (43)$$

This gives the more generalized version of the expression relating the teleportation fidelities of three initial resource states with the final remotely prepared states.

Next we consider a more complicated situation where we have $(g + 1)$ entangled states distributed among hypothetical parties in $g + 2$ nodes. These entangled states are shared between consecutive nodes. We consider two types of measurement namely simultaneous and consecutive measurements M_1, M_2, \dots, M_g at g nodes between the initial and the final nodes. As we have seen in the previous section that both of these measurements create entanglement between the first and final node. Here we prove a theorem, quite analogous to previous theorems relating the teleportation capability of the resource states with the final state obtained as a result of swapping in the process of remote entanglement distribution (RED).

Theorem II: If we start with $(g + 1)$ entangled states in the most general form, $|\psi\rangle_{12}, |\psi\rangle_{23}, \dots, |\psi\rangle_{(g+1)(g+2)}$, between the nodes (1, 2), (2, 3), ..., $(g + 1, g + 2)$ with respective teleportation fidelities $F(|\psi\rangle_{12}), F(|\psi\rangle_{23}), \dots, F(|\psi\rangle_{(g+1)(g+2)})$,

then the teleportation fidelity of the state $|\chi^{r_1 h_1, r_2 h_2, \dots, r_g h_g}\rangle_{1(g+2)}$ is given by

$$3F(|\chi^{r_1 h_1, r_2 h_2, \dots, r_g h_g}\rangle_{1(g+2)}) - 2 = \frac{\prod_{i=1}^g F_{r_i h_i}}{2^g M_{r_1 h_1, r_2 h_2, \dots, r_g h_g}} [3F(|\psi\rangle_{12}) - 2] [3F(|\psi\rangle_{23}) - 2] \dots [3F(|\psi\rangle_{(g+1)(g+2)}) - 2]$$

Proof: Here we start with $(g + 1)$ entangled states in the most general form, $|\psi\rangle = \sum_{i_k, j_k=0} a_{i_k j_k} |i_k j_k\rangle$, where k denotes the index for the number of entangled states and varies from 0 to g . Here for a fixed k , $a_{i_k j_k}$ denotes the corresponding coefficients of the given entangled state. Then we create an entangled state between the qubits at the nodes 1 and $g + 2$ by entanglement swapping. This process is carried out either by simultaneous measurements M_1, M_2, \dots, M_g at the nodes 2, 3, 4, ..., $g + 1$ respectively or by sequential measurements at the intermediate nodes.

From the previous section, we had already seen that the concurrences for the initial states and final state for both the types of measurements are related by,

$$C(|\chi^{r_1 h_1, r_2 h_2, \dots, r_g h_g}\rangle_{1(g+2)}) = \frac{\prod_{i=1}^g F_{r_i h_i}}{2^g M_{r_1 h_1, r_2 h_2, \dots, r_g h_g}} C(|\psi\rangle_{12}) \cdot C(|\psi\rangle_{23}) \dots C(|\psi\rangle_{(g+1)(g+2)}). \quad (44)$$

The superscripts $r_i h_i, i = 1, 2, \dots, g$ comes from the measurement of g parties (i.e., if their measurement results are

$$|\phi^{r_1 h_1}\rangle \otimes |\phi^{r_2 h_2}\rangle \otimes \dots \otimes |\phi^{r_g h_g}\rangle = \bigotimes_{i=1}^g |\phi^{r_i h_i}\rangle.$$

Now by substituting the values of the concurrences in terms of the teleportation fidelities in the above relation we finally obtain the relation involving the teleportation fidelities of the initial resource states with the final remotely prepared state as

$$3F(|\chi^{r_1 h_1, r_2 h_2, \dots, r_g h_g}\rangle_{1, g+2}) - 2 = \frac{\prod_{i=1}^g F_{r_i h_i}}{2^g M_{r_1 h_1, r_2 h_2, \dots, r_g h_g}} [3F(|\psi\rangle_{12}) - 2] [3F(|\psi\rangle_{23}) - 2] \dots [3F(|\psi\rangle_{(g+1)(g+2)}) - 2]. \quad (45)$$

Relations On Super Dense Coding Capacity In Remote Entangled Distribution (RED)

Quantum super dense coding involves in sending of classical information from one sender to the receiver when they

are sharing a quantum resource in form of an entangled state. More specifically superdense coding is a technique used in quantum information theory to transmit classical information by sending quantum systems [23]. In the simplest case, Alice wants to send Bob a binary number $x \in \{00, 01, 10, 11\}$. She picks up one of the unitary operators I, X, Y, Z according to x she has chosen and applies the transformation on her multiqubit (the first multiqubit of the Bell state shared by them). Alice sends her multiqubit to Bob after one of the local unitaries are applied. The state obtained by Bob will be one of the four basis vectors, so he performs the measurement in the Bell basis to obtain two bits of information. It is quite well known that if we have a maximally entangled state in $H_d \otimes H_d$ as our resource, then we can send $2 \log d$ bits of classical information. In the asymptotic case, we know one can send $\log d + S(\rho)$ amount of bit when one considers non-maximally entangled state as resource [30–34]. It had been seen that the number of classical bits one can transmit using a non-maximally entangled state in $H_d \otimes H_d$ as a resource is $(1 + p_0 \frac{d}{d-1}) \log d$, where p_0 is the smallest Schmidt coefficient. However, when the state is maximally entangled in its subspace then one can send up to $2 * \log(d - 1)$ bits [35].

In particular super dense coding capacity for a mixed state ρ_{AB} is defined by

$$C_{AB} = \log_2 d + S(\rho_B) - S(\rho_{AB}), \quad (46)$$

where $\rho_B = \text{tr}_A[\rho_{AB}]$. Here we note that the expression $S(\rho_B) - S(\rho_{AB})$ can either be positive or negative. If it is positive then one can use the shared state to transfer bits greater than the classical limit of $\log_2 d$ bits. For pure states, $S(\rho_{AB}) = 0$, then the superdense coding capacity is given by,

$$C_{AB} = \log_2 d + S(\rho_B) = \log_2 d + E(\rho_{AB}), \quad (47)$$

where the entanglement entropy $E(\rho_{AB})$ of the state ρ_{AB} is nothing but the von neumann entropy of the sub system of the reduced subsystem ρ_B . For separable state the super dense coding capacity is $\log_2 d$.

In this subsection we find how the superdense coding capacities of the resource states are related with the superdense coding capacity of the entangled state obtained as a result of entanglement swapping. More precisely in a quantum network we start with various resource states connecting the consecutive nodes, and we want to send the classical information from the initial node to the final node. One way of doing it is by first creating an entangled states between the end nodes, by doing measurements on the intermediate nodes [20, 21]. Indeed this is the process of entanglement swapping and falls into the broader class of remote entanglement distribution (RED). Once the entangled state is created, we

can send the classical information. Here in this subsection we give an important relationship involving the superdense coding capacities (capacities of sending classical information) of the resource entangled states with that of the final entangled state.

Here we consider only the simplest situation where we have two resource states at our disposal and we want to send classical information from one node to another which are not initially entangled. Let us once again begin with a situation where two parties Alice, Bob sharing an entangled state $|\psi\rangle_{12} = \sum_i \lambda_i |ii\rangle$ between them, where as Bob and Charlie share another state $|\psi\rangle_{23} = \sum_j \mu_j |jj\rangle$ (where $\lambda_i, \mu_j, (i, j = 0, 1, \dots, d)$ are the Schmidt coefficients, satisfying $\sum_i \lambda_i^2 = 1, \sum_j \mu_j^2 = 1$) with each other [29]. There arises three situations depending upon the choice of the schmidt coefficients of the resource states. Each of these relations involving super dense coding capacity of the initial and final entangled state follows from the results we have already obtained for entanglement entropy.

Case I:

First of all we consider the case when both the resource states are maximally entangled. In other words this refers to the situation when all the Schmidt coefficients are equal to $\frac{1}{\sqrt{d}}$. Then the super dense coding capacity of the resource state is related with the super dense coding capacity of the remotely prepared entangled states $|\chi^{rh}\rangle_{13}$ (where r, h are the indices indicating the measurement outcomes) as,

$$\mathcal{C}(|\chi^{rh}\rangle_{13}) = \mathcal{C}(|\psi\rangle_{12}) = \mathcal{C}(|\psi\rangle_{23}). \quad (48)$$

Case II:

In this particular case we consider the situation when one of the entangled state is maximally entangled and the rest is non maximally entangled i.e $\lambda_i = \frac{1}{\sqrt{d}}, \mu_j \neq \frac{1}{\sqrt{d}}$, then we have,

$$\mathcal{C}(|\chi^{rh}\rangle_{13}) = \mathcal{C}(|\psi\rangle_{23}). \quad (49)$$

In the other case $\lambda_i \neq \frac{1}{\sqrt{d}}, \mu_j = \frac{1}{\sqrt{d}}$, then we have,

$$\mathcal{C}(|\chi^{rh}\rangle_{13}) = \mathcal{C}(|\psi\rangle_{12}). \quad (50)$$

Case III:

Finally, we consider the case when both the entangled states are not maximally entangled i.e $\lambda_i \neq \frac{1}{\sqrt{d}}, \mu_j \neq \frac{1}{\sqrt{d}}$, then the super dense coding capacity of the swapped state is given by,

$$\mathcal{C}(|\chi^{rh}\rangle_{13}) < \max[\mathcal{C}(|\psi\rangle_{12}), \mathcal{C}(|\psi\rangle_{23})]. \quad (51)$$

VI CONCLUSION

In a nutshell, here in this work, we established an important relationship connecting the fidelities of teleportation of the resource states with the fidelity of the final state obtained as a result of entanglement swapping. Similarly we also connected the superdense coding capacities of the resource states with that of the final state. All these relations are very much important and relevant in the context of quantum networking. These relations actually determine the amount of information both classical and quantum, one can send from one node to a desired node in a quantum network. In other words, in an arbitrary network when two nodes are not connected, our result shows how much information both quantum and classical can be sent from one node to other. In fact the amount of transferable information depends on the capacities of the inter connecting entangled resources. Depending upon the inter connecting entangled resources, we can choose the optimal path in a quantum network to send the maximal possible information.

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